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MAXIMAL THEOREMS FOR SOME ORTHOGONAL SERIES. I

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The idea of deducing properties of an orthogonal expansion with respect to one orthonormal sequence from the orthogonal expansion with respect to another orthonormal sequence or of relating these expansions has long been common. For instance, if $f \sim \sum_n \hat{f}(n) u_n(x)$ is the orthogonal development, the Marcel Riesz theorem can be reformulated as: let $\{a_n\}$ be a sequence of real numbers; then $f \sim \sum_n a_n \cos nx$ for some $f \in L^p(0, \pi)$, $1 < p < \infty$, if and only if $\phi \sim \sum_n a_n \sin nx$ for some $\phi \in L^p(0, \pi)$. In other directions there is a large body of equiconvergence or equisummability theorems. Recently, several authors have employed such ideas to obtain important results for expansions with respect to ultraspherical and Jacobi polynomials either by analogy with the corresponding results for Fourier series [14] or by "transplantation" of orthonormal sequences (cf. [2], [4]). In this paper a more axiomatic treatment is given.

When $\{u_n\}$, $\{v_n\}$ are sequences of real-valued functions on $(0, \pi)$ not necessarily orthonormal or identical, theorems of maximal type are proved relating the properties of kernels:

$$K_k(x, t) = \sum_n r_{nk} e^{inx} e^{-int}, \qquad \mathscr{K}_k(x, t) = \sum_n r_{nk} u_n(x) v_n(t).$$

The coefficients $\{r_{nk}\}$ could arise from a summability method, for instance. Under five simple conditions on $\{u_n\}$, $\{v_n\}$ the properties of

$$\int_{-\pi}^{\pi} f(t) K_k(x, t)\, dt, \qquad f \in L^p(-\pi, \pi)(^1),$$

i.e. for ordinary Fourier series, can be carried over to $\int_0^\pi \phi(t) \mathscr{K}_k(x, t)\, dt$, $\phi \in L^p(0, \pi)$, provided $p \neq 1, \infty$ (§1). Although the case $p = 1$ is more difficult (perhaps not surprisingly), a slight strengthening of one of the five conditions is all that is required. If now $\{u_n\}$, $\{v_n\}$ are complete and orthonormal, properties of the orthogonal expansions with respect to $\{u_n\}$, $\{v_n\}$ can be read off from the corresponding results for Fourier series (§2). Typical results are:

(i) *Let $\{a_n\}$ be a sequence of real numbers. Then $f \sim \sum a_n \cos nx$ for some $f \in L^p(0, \pi)$ if and only if $\phi \sim \sum_n a_n u_n(x)$ for some $\phi \in L^p(0, \pi)$, $p \neq 1, \infty$.*

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(1) All functions considered are real valued.

(ii) *Let* $\hat{f}(n)=\int_0^\pi f(t)v_n(t)\,dt$ *be the coefficients of* $f\in L^p(0,\pi)$ *with respect to* $\{v_n\}$, $p\neq 1,\infty$. *Then* $\sum_n \hat{f}(n)u_n(x)$ *converges a.e. on* $(0,\pi)$ *and*

$$\left\| \sup_N \left| \sum_0^N \hat{f}(n)u_n(x)\right| \right\|_p \leqq A_p\|f\|_p.$$

Note that in (ii) the sequences need not be identical. In (i) use of the Marcel Riesz theorem in some form or other seems unavoidable, so, unfortunately, (i) cannot be used as an alternative proof of the Marcel Riesz theorem; result (ii) requires the full strength of the Carleson-Hunt theorem for Fourier series [7], [12].

All five conditions are satisfied, for example, by the classical orthonormal sequences on $(0,\pi)$: Fourier-Bessel functions, Fourier-Dini functions, ultraspherical polynomials and Jacobi polynomials. Thus, in result (ii), we could form the coefficients $\{\hat{f}(n)\}$ with respect to Jacobi polynomials $\{v_n\}$ and then the series $\sum_n \hat{f}(n)u_n(x)$ with Fourier-Dini functions $\{u_n\}$. Many of the results of Askey-Wainger [4], Askey [2] and Muckenhoupt-Stein [14] also are included in this axiomatic approach. Elsewhere we shall show that the five conditions are satisfied by two large classes of Sturm-Liouville systems [11]. Basically, what is required is a good enough 2-term asymptotic estimate, say a 2-term Hilb-type estimate.

At each stage of the proof of these "transplantation" theorems we have attempted to single out the conditions which must be imposed on $\{u_n\}$, $\{v_n\}$ to ensure the validity of the results. Obviously some conditions are required since no general orthogonal series has properties so closely analogous to Fourier series. For a discussion of the concepts used in this paper without formal definition see Edwards [9], Zygmund [24].

The proof given here of Theorem 1—the main result of the paper—is based in part on the ideas used by Askey in [2], and I wish to thank Professor Askey for showing me his paper in prepublication form.

1. **Transplantation theorem.** In this section we shall prove four theorems of transplantation type (cf., for instance, Askey [1]) relating the properties of a general kernel with those of the corresponding kernel for Fourier series. The following five conditions will be imposed on $\{u_n\}$:

(P.1) *For some constant* A $\sup_{0<x<\pi}|u_n(x)|\leqq A$, $n=0,1,\ldots$.

(P.2) *There exist functions* $X_1,\ldots,X_4\in L^\infty(0,\pi/2)$ *such that*

$$\begin{aligned} u_n(x) &= X_1(x)\cos nx + X_2(x)\sin nx \\ &\quad + (1/nx)\{X_3(x)\cos nx + X_4(x)\sin nx\} + O((nx)^{-2}) \end{aligned}$$

uniformly for $x\in(1/n,\pi/2)$.

The next two conditions estimate the difference $\Delta(u_n)=u_n-u_{n+1}$. In many applications the expressions for $x\in(0,1/n)$, $x\in(1/n,\pi/2)$ seem to have to be estimated separately but typical conditions are

(P.3) *There exist functions* $X_1', \ldots, X_4' \in L^\infty(0, \pi/2)$ *such that*

$$\Delta(u_n) = x\{X_1'(x)\cos nx + X_2'(x)\sin nx\} + (1/n)\{X_3'(x)\cos nx + X_4'(x)\sin nx\} + O(n^{-2}x^{-1})$$

uniformly for $x \in (1/n, \pi/2)$.

(P.4) *There is a function* $X \in L^\infty(0, \pi/2)$ *for which*

$$\Delta(u_n) = (1/n)X(x)u_n(x) + (n^{-2}+x)O(1)$$

uniformly in $(0, 1/n)$.

The final condition enables us to "change variables".

(P.5) *There is a sequence* $\{U_n\}$ *satisfying* (P.1), ..., (P.4) *such that* $u_n(\pi - x) = (-1)^n U_n(x) + O(n^{-2})$ *uniformly in* $x \in (0, \pi/2)$.

Actually, it will be enough that (P.1) is satisfied for all $n \geqq 0$ while (P.2), ..., (P.5) hold for $n > N$, say (cf. Remarks 1.5). Usually, in condition (P.5) the sequence $\{U_n\}$ differs from $\{u_n\}$ as in Jacobi polynomials or Fourier-Bessel functions, for example.

Now let $\{u_n\}$, $\{v_n\}$ be sequences of (real-valued) functions, not necessarily the same, both of which satisfy (P.1), ..., (P.5). Corresponding to sequences $\{r_{nk}\}$, $k = 0, 1, \ldots$, of real numbers with $\sum_{n=0}^\infty |r_{nk}| < \infty$, we define kernels $K_k(x, t)$, $\mathscr{K}_k(x, t)$ by

$$K_k(x, t) = \sum_{n=0}^\infty r_{nk}e^{inx}e^{-int}, \qquad \mathscr{K}_k(x, t) = \sum_{n=0}^\infty r_{nk}u_n(x)v_n(t).$$

In this section it need not be assumed that $\{u_n\}$, $\{v_n\}$ consist of orthogonal functions nor that $\{u_n\}$, $\{v_n\}$ are complete in $L^p(0, \pi)$. Under the conditions

I. $|r_{nk}| \leqq B$, $n, k \geqq 0$,

II. $\sum_n |\Delta(r_{nk})| \leqq B$, $k \geqq 0$,

(B constant) we shall prove:

THEOREM 1. *Suppose the operator*

$$f \to S^*f(x) = \sup_k \left| \int_{-\pi}^{\pi} f(t)K_k(x, t)\, dt \right|$$

is of weak type (p, p) *with respect to* $L^p(-\pi, \pi)$ *for some* p, $1 < p < \infty$. *Then the operator*

$$\phi \to T^*\phi(x) = \sup_k \left| \int_0^{\pi} \phi(t)\mathscr{K}_k(x, t)\, dt \right|$$

is of weak type (p, p) *with respect to* $L^p(0, \pi)$ *for this same* p.

There is an entirely analogous theorem with "strong type (p, p)" replacing "weak type (p, p)". When $p = 1$ it is unlikely that Theorem 1 is true in general if only conditions (P.1), ..., (P.5) are assumed. However, we can prove, using a virtually identical proof, THEOREM 2. *If the operator* $f \to S^*f$ *is of weak type* (1, 1) *then*

$$m\{x \in (0, \pi) : T^*\phi > s\} \leqq \frac{\text{const}}{s} \{\|\phi \log^+ |\phi| \|_1 + 1\}, \qquad s > 0,$$

whenever $\phi \in L \log L^+(0, \pi)$.

No result for strong type (1, 1) can be expected because of the failure of the Marcel Riesz theorem; indeed, orthogonal expansions even with respect to $\{(2/\pi)^{1/2} \cos nx\}$, $\{(2/\pi)^{1/2} \sin nx\}$ behave markedly differently for $p=1$ in contrast to the results proved in this paper. Further confirmation is provided by an example of Askey and Wainger [3, p. 217]. For comments on the necessity of conditions I, II and applicability see Remarks 2.1.

By imposing a stronger condition than (P.2) we can, in fact, get the weak type (1, 1) and the expected strong type result:

(P.2)′ *There exist functions* $X_1, X_2 \in L^\infty(0, \pi/2)$ *and constants* α, β *such that*

$$u_n(x) = X_1(x) \cos (\lambda_n x + \beta) + (1/nx) X_2(x) \sin (\lambda_n x + \beta) + O((nx)^{-2}), \qquad \lambda_n = n + \alpha,$$

uniformly in $(1/n, \pi/2)$.

Obviously (P.2)′ implies (P.2). In practice (P.2)′ is easily derived from the corresponding expression in which

$$\lambda_n = n + \alpha + a/n + O(n^{-2}), \qquad n \to \infty, \tag{1}$$

where α, a are constants.

Suppose now that (P.1), (P.2)′, (P.3), ..., (P.5)[2] are satisfied by $\{u_n\}$.

THEOREM 3. *If the operator* $f \to S^*f$ *is of weak type* (1, 1) *then the operator*

$$\phi \to \mathscr{T}^*\phi(x) = \sup_k \left| \int_0^\pi \phi(t) \left\{ \sum_n r_{nk} u_n(x) u_n(t) \right\} dt \right|$$

is of weak type (1, 1) *with respect to* $L^1(0, \pi)$.

THEOREM 4. *Suppose the operator* $f \to S^*f$ *satisfies*

$$\|S^*f\|_1 \leqq A\{\|f \log^+ |f|\|_1 + 1\},$$

whenever $f \in L \log L^+(-\pi, \pi)$. *Then, for some constant* A',

$$\|\mathscr{T}^*\phi\|_1 \leqq A'\{\|\phi \log^+ |\phi|\|_1 + 1\}$$

provided $\phi \in L \log L^+(0, \pi)$.

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(2) For theorems requiring (P.2)′, (P.5) has to have a stronger form in which the sequence $\{U_n\}$ satisfies (P.2)′ as well as (P.1), (P.3), (P.4).

Some simple preliminary results will make the proof of Theorems 1–4 run more smoothly. Set

$$C_n(x) = \sum_0^n \cos mx, \qquad S_n(x) = \sum_1^n \sin mx$$

and

$$D_{nk}(x) = \sum_0^n r_{mk} \cos mx, \qquad E_{nk}(x) = \sum_1^n r_{mk} \sin mx.$$

Then $C_n(x)$, $S_n(x) = O(x^{-1})$ or $O(n)$ uniformly in n or x respectively, $x \in (0, \pi)$. Also, there exist functions X_1, $X_2 \in L^\infty(0, \pi)$ such that

$$C_n(x) = (1/x)\{X_1(x) \cos nx + X_2(x) \sin nx\}, \qquad x \in (0, \pi), \tag{2}$$

with a corresponding expression for $S_n(x)$. Summation by parts of $D_{nk}(x)$, $E_{nk}(x)$ shows that

$$D_{nk}(x) = O(x^{-1}), \qquad E_{nk}(x) = O(x^{-1}), \qquad x \in (0, \pi)$$

uniformly in n, k whenever conditions I, II are satisfied. By changing variables (or summing by parts) we deduce that all the series

$$\sum_0^n (-1)^m \cos mx, \ldots = O(1), \qquad x \in (0, 3\pi/4), \tag{3}$$

uniformly in n, x and k.

It is well known that the operators

$$\begin{gathered} \phi \to \frac{1}{x} \int_0^x \phi(t)\, dt, \qquad \phi \to \int_x^\pi (\phi(t)/t)\, dt, \\ \phi \to \phi^*(x) = \sup_h \frac{1}{|h|} \int_x^{x+h} |\phi(t)|\, dt \end{gathered} \tag{4}$$

are all of weak type (p, p) with respect to $L^p(0, \pi)$ for all $1 \leqq p < \infty$, (cf. Zygmund [24]). Now, using the Marcel Riesz theorem for the (principal-valued) integral $\int_{-\pi}^{\pi} f(t) \cot \frac{1}{2}(x-t)\, dt$, we can soon check that

$$\phi \to \int_0^\pi \frac{\phi(t)}{\cos t - \cos x} \begin{Bmatrix} \sin x \\ \sin t \end{Bmatrix} dt \tag{5}$$

is of weak type (p, p) with respect to $L^p(0, \pi)$, $1 \leqq p < \infty$, where, of course, the integral is taken in the principal-value sense.

In the following four lemmas the notation

$$\begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} \begin{Bmatrix} \cos nt \\ \sin nt \end{Bmatrix}$$

will mean that all four possible combinations are allowed but, in any given series, the same combination must occur for all n.

1.1. Lemma. *If the operator $f \to S^*f$ is of weak type (p, p) for some p, $1 \leqq p < \infty$, then each of the operators*

$$\phi \to \sup_k \left| \int_0^\pi \phi(t) \left[\sum_{n=0}^\infty r_{nk} \begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} \begin{Bmatrix} \cos nt \\ \sin nt \end{Bmatrix} \right] dt \right| \tag{6}$$

is of weak type (p, p) with respect to $L^p(0, \pi)$ for this same p.

Proof. By hypothesis, $f \to S^*f$ and hence also

$$f \to \sup_k \left| \int_{-\pi}^\pi f(t) K_k(-x, t)\, dt \right|$$

is of weak type (p, p) (with respect to $L^p(-\pi, \pi)$). Now, given $\phi \in L^p(0, \pi)$, extend ϕ to an even function f on $(-\pi, \pi)$; then

$$\int_0^\pi \phi(t) \left\{ \sum_{n=0}^\infty r_{nk} \cos nx \cos nt \right\} dt = \frac{1}{2} \int_{-\pi}^\pi f(t) \left\{ \sum_{n=0}^\infty r_{nk} \cos nx\, e^{-int} \right\} dt.$$

Since $\sum_n r_{nk} \cos nx\, e^{-int} = \frac{1}{2}[K_k(x, t) + K_k(-x, t)]$, this establishes the first of the assertions. When, for instance, $\sin nt$ occurs in (6), ϕ has to be extended to an odd function on $(-\pi, \pi)$, suitably modifying ϕ at 0 if necessary. The remaining assertions follow easily.

1.2. Lemma. *Suppose the sequences $\{r_{nk}\}$ satisfy conditions* I, II *and that $f \to S^*f$ is of weak type (p, p). Then each of the operators*

$$\phi \to \sup_k \left| \int_{x/2}^{2x} \phi(t) \left[\sum_n r_{nk} \begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} \begin{Bmatrix} \cos nt \\ \sin nt \end{Bmatrix} \right] dt \right| \tag{7}$$

is of weak type (p, p) with respect to $L^p(0, \pi/2)$, ϕ being extended by 0 *outside of* $(0, \pi/2)$.

Proof. In view of Lemma 1.1 it is enough to consider, say, the operators

$$\phi \to \sup_k \left| \int_0^{x/2} \phi(t) \left\{ \sum_n r_{nk} \cos nx \cos nt \right\} dt \right|, \tag{8}$$

$$\phi \to \sup_k \left| \int_{2x}^{\pi/2} \phi(t) \left\{ \sum_n r_{nk} \cos nx \cos nt \right\} dt \right|. \tag{9}$$

Summation by parts in (i) gives([3])

$$\frac{1}{2} \int_0^{x/2} \phi(t) \left[\sum_n \{C_n(x-t) + C_n(x+t)\}\, \Delta(r_{nk}) \right] dt = O\left(\frac{1}{x} \int_0^{x/2} |\phi| \right)$$

$$= O(\phi^*(x)) + O\left(\frac{1}{x} \int_0^x |\phi| \right)$$

([3]) If $\{r_{nk}\}$ contains only finitely many nonzero terms (for fixed k) a term $O(\int_0^{\pi/2} |\phi|)$ has to be added. This is uniform in k by condition I and so does not affect the boundedness of the operator. A similar comment applies also in all subsequent summations by parts.

uniformly in k using condition II. By a similar argument, (9) is

$$O(\phi^*(x))+O\left(\int_x^{\pi/2} (|\phi|/t)\right),$$

also uniformly in k. Lemma 1.1 together with the properties of (4) completes the proof.

1.3. LEMMA. *Suppose the sequences* $\{r_{nk}\}$ *satisfy conditions* I, II *and that, for* $\phi \in L^p(\pi/2, \pi)$,

$$U^*\phi(x) = \sup_k \left| \int_{x/2}^{2x} \phi(\pi-\eta)\left[\sum_n r_{nk} \begin{Bmatrix} \cos nx \\ \sin nx \end{Bmatrix} \begin{Bmatrix} \cos n(\pi-\eta) \\ \sin n(\pi-\eta) \end{Bmatrix}\right] d\eta \right|,$$

where $x, \eta \in (0, \pi/2)$. *Then, for each* p, $1 \leqq p < \infty$,

$$m\{x \in (0, \pi/2) : U^*\phi > s\} \leqq \frac{\text{const}}{s^p}\left(\int_{\pi/2}^{\pi} |\phi(x)|^p\, dx\right), \qquad s > 0, \tag{10}$$

ϕ *being extended by* 0 *outside* $(\pi/2, \pi)$.

Proof. Consider first the operator

$$\phi \to U_1^*\phi(x) = \sup_k \left| \int_{\pi/2}^{\pi} \phi(t)\left\{\sum_n r_{nk} \cos nx \cos nt\right\} dt \right|.$$

Now, a routine calculation shows that $\frac{1}{2}\{C_n(x-t)+C_n(x+t)\}$ is

$$O(1)-\frac{\sin nt \cos nx \sin t - \sin nx \cos nt \sin x}{\cos t - \cos x}.$$

Consequently, after summation by parts, $U_1^*\phi$ becomes $O(\int_{\pi/2}^{\pi} |\phi|)$ together with

$$\sup_k \left| \int_{\pi/2}^{\pi} \phi(t)\left[\sum_n \Delta(r_{nk})\left\{\frac{\sin nt \cos nx \sin t - \sin nx \cos nt \sin x}{\cos t - \cos x}\right\}\right] dt \right|$$

where in the displayed term, x ranges over $(0, \pi/2)$. But then $\cos t - \cos x$ has constant sign and so the term is

$$O\left(\left|\int_{\pi/2}^{\pi} \frac{|\phi(t)|}{\cos t - \cos x} \begin{Bmatrix} \sin t \\ \sin x \end{Bmatrix} dt\right|\right)$$

uniformly in k, the integrals being taken in the principal-value sense. Hence, by the Riesz theorem for (5),

$$m\{x \in (0, \pi/2) : U_1^*\phi > s\} \leqq \frac{\text{const}}{s^p}\left(\int_{\pi/2}^{\pi} |\phi(x)|^p\, dx\right), \qquad s > 0. \tag{11}$$

Now consider say

$$\int_0^{x/2} \phi(\pi-\eta)\left\{\sum_n (-1)^n r_{nk} \cos nx \cos n\eta\right\} d\eta, \qquad x \in (0, \pi/2). \tag{12}$$

Using conditions I, II and (3) since $x \in (0, \pi/2)$, $\eta \in (0, \pi/4)$, we deduce that (12) is

$$\int_0^{x/2} \phi(t)\Big\{\sum_n O(1)\,\Delta(r_{nk})\Big\}\,dt = O\Big(\int_{\pi/2}^{\pi} |\phi|\Big)$$

uniformly in k. On the other hand, the integral $\int_{2x}^{\pi/2} (\cdot)\,d\eta$ corresponding to (12) is 0 unless $x<\pi/4$ and so, by the same argument, this integral too is $O(\int_{\pi/2}^{\pi} |\phi|)$. Combination of these estimates with (11) establishes (10).

An analogous argument proves

1.4. Lemma. *For $\phi \in L^p(0, \pi/2)$, $\xi=\pi-x \in (0, \pi/2)$ set*

$$V^*\phi(x) = \sup_k \left|\int_{\xi/2}^{2\xi} \phi(t)\Big[\sum_n r_{nk}\begin{Bmatrix}\cos nx\\ \sin nx\end{Bmatrix}\begin{Bmatrix}\cos nt\\ \sin nt\end{Bmatrix}\Big]\,dt\right|.$$

Then, under conditions I, II, *for each p, $1 \leqq p < \infty$,*

$$m\{x \in (\pi/2, \pi) : V^*\phi > s\} \leqq \frac{\text{const}}{s^p}\Big(\int_0^{\pi/2} |\phi(x)|^p\,dx\Big),$$

ϕ being extended by 0 *outside* $(0, \pi/2)$.

Proof of Theorem I. It is enough to show that, under the given conditions,

(13) $$m\{x \in (0, \pi) : T^*\phi > s\} \leqq A_p\{(1/s)\|\phi\|_p\}^p, \qquad s > 0,\ \phi \in L^p(0, \pi),$$

where $\|(\cdot)\|_p$ denotes the usual L^p-norm on $(0, \pi)$. The proof proceeds in two stages.

Stage 1. Suppose $x, t \in (0, \pi/2)$ and let $\chi=2[1/x]+1$; also, extend any function in $L^p(0, \pi/2)$ by 0 outside $(0, \pi/2)$.

(i) $x/2 \leqq t \leqq 2x$. Set

$$I_1 = \int_{x/2}^{2x} \phi(t)\Big\{\Big(\sum_0^{\chi-1} + \sum_{\chi}^{\infty}\Big) r_{nk}u_n(x)v_n(t)\Big\}\,dt = J_{11}+J_{12}.$$

By condition I and (P.1),

$$J_{11} = O\Big(\frac{1}{x}\int_{x/2}^{2x} |\phi|\,dt\Big) = O(\phi^*(x))$$

uniformly in k. To handle J_{12}, replace $u_n(x)$, $v_n(t)$ by their asymptotic estimates (P.2). Then J_{12} is dominated by terms of the form

$$O\Big(\Big|\int_{x/2}^{2x} \phi(t)X_1(t)\Big\{\sum_{\chi}^{\infty} r_{nk}\cos nx\cos nt\Big\}\,dt\Big|\Big)$$
$$+O\Big(\Big|\int_{x/2}^{2x} \phi(t)X_2(t)\Big\{\sum_{\chi}^{\infty} \frac{1}{nx} r_{nk}\cos nx\cos nt\Big\}\,dt\Big|\Big)$$
$$+O\Big(\Big|\int_{x/2}^{2x} \phi(t)X_3(t)\Big\{\sum_{\chi}^{\infty} \frac{1}{nt} r_{nk}\cos nx\cos nt\Big\}\,dt\Big|\Big)$$
$$+O\Big(\Big|\int_{x/2}^{2x} \phi(t)X_4(t)\Big\{\sum_{\chi}^{\infty} (nx)^{-2}\Big\}\,dt\Big|\Big) = J_{13}+\cdots+J_{16}$$

where $X_1, \ldots, X_4 \in L^\infty(0, \pi/2)$. Clearly

$$J_{16} = O\left(\frac{1}{x}\int_{x/2}^{2x} |\phi| \, dt\right) = O(\phi^*(x))$$

uniformly in k. By allowing the infinite series in J_{13} to range from 0 to ∞ instead of χ to ∞ we obtain integrals of the form (7) and an error term analogous to J_{11}; consequently J_{13} is dominated by terms of the form

$$O\left(\left|\int_{x/2}^{2x} \phi(t)X_1(t)\left\{\sum_n r_{nk} \cos nx \cos nt\right\} dt\right|\right) + O(\phi^*(x)).$$

Now consider J_{14}, or more briefly[4],

$$\int_{x/2}^{2x} \phi(t)X_2(t)\left\{\sum_{\chi}^{\infty} \frac{1}{nx} r_{nk} \cos nx \cos nt\right\} dt. \tag{14}$$

Integration by parts yields (the series is absolutely convergent)

$$\left[\Phi(t)\left\{\sum_{\chi}^{\infty} \frac{1}{nx} r_{nk} \cos nx \cos nt\right\}\right]_{x/2}^{2x} + \frac{1}{x}\int_{x/2}^{2x} \Phi(t)\left\{\sum_{\chi}^{\infty} r_{nk} \cos nx \sin nt\right\} dt = J_{17} + J_{18},$$

where we have written $\Phi(t) = \int_0^t \phi(\tau)X_2(\tau)\, d\tau$. If we sum by parts the series in J_{17} we obtain a possible term $O(\phi^*(x))$ (see [3]), a term

$$\frac{1}{2}\left[\frac{1}{x}\int_0^{x/2} \phi(t)X_2(t)\left(\sum_{\chi}^{\infty} \Delta\left(\frac{1}{n}\right)\left\{D_{nk}\left(\frac{x}{2}\right) + D_{nk}\left(\frac{3x}{2}\right)\right\}\right) dt\right] = O\left(\sum_{\chi}^{\infty} n^{-2}x^{-1}\right)\phi^*(x) = O(\phi^*(x))$$

uniformly in k and a similar term when $x/2$ is replaced by $2x$ (or $\pi/2$ if $2x > \pi/2$). To handle J_{18} write the integral as

$$\int_{x/2}^{2x} \frac{1}{t}\, \Phi(t)\left\{\sum_{\chi}^{\infty} r_{nk} \cos nx \sin nt\right\} dt \tag{15}$$

$$-\frac{1}{x}\int_{x/2}^{2x} \frac{1}{t}\,(x-t)\Phi(t)\left\{\sum_{\chi}^{\infty} r_{nk} \cos nx \cos nt\right\} dt. \tag{16}$$

After summation by parts we can soon show that (16) is

$$O\left(\frac{1}{x}\int_{x/2}^{2x} \frac{1}{t}\, |\Phi(t)| \, dt\right) = O\left(\frac{1}{x}\int_{x/2}^{2x} \left\{\frac{1}{t}\int_0^t |\phi|\right\} dt\right), \tag{17}$$

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[4] The upper limit is taken as $\pi/2$ if $2x > \pi/2$.

while (15) is again of the form (7). A similar argument with Φ replaced by

$$\left(\int_t^{\pi/2} \frac{1}{\tau}\,\phi(\tau)X_2(\tau)\,d\tau\right)$$

gives analogous estimates for J_{15}. Collecting together all these estimates we deduce that

$$m\left\{x \in \left(0, \frac{\pi}{2}\right) : \sup_k |I_1| > s\right\} \leqq A'_p\left(\frac{1}{s}\,\|\phi\|_p\right)^p, \qquad \phi \in L^p\left(0, \frac{\pi}{2}\right).$$

(ii) $t < x/2$. Set

$$I_2 = \int_0^{x/2} \phi(t)\left\{\left(\sum_0^{\chi-1} + \sum_\chi^\infty\right) r_{nk} u_n(x) v_n(t)\right\} dt = J_{21} + J_{22}.$$

As before

$$J_{21} = O\left(\frac{1}{x}\int_0^{x/2} |\phi|\,dt\right) = O(\phi^*(x)),$$

uniformly in k. If, in J_{22}, $u_n(x)$ is replaced by its estimate (P.2) we see that J_{22} is dominated by terms of the form

$$O\left(\left|\int_0^{x/2} \phi(t)X_1(t)\left\{\sum_\chi^\infty r_{nk}\cos nx\, v_n(t)\right\} dt\right|\right)$$
$$+O\left(\left|\int_0^{x/2} \phi(t)X_2(t)\left\{\sum_\chi^\infty \frac{1}{nx}\, r_{nk}\cos nx\, v_n(t)\right\} dt\right|\right)$$
$$+O\left(\int_0^{x/2} |\phi(t)|\left\{\sum_\chi^\infty (nx)^{-2}\right\} dt\right) = J_{23} + \cdots + J_{25},$$

with $X_1, X_2 \in L^\infty(0, \pi/2)$. As before, uniformly in k,

$$J_{25} = O\left(\frac{1}{x}\int_0^{x/2} |\phi|\right) = O(\phi^*(x)).$$

Estimates for J_{23}, J_{24} are obtained by careful use of summation by parts arguments. Now, by condition II, if $\tau = [1/t]+1$,

$$\sum_\chi^\infty r_{nk}\cos nx\, v_n(t) = \sum_\chi^\infty C_n(x)\,\Delta(r_{nk}v_n) \tag{18}$$
$$= O(x^{-1}) + \left(\sum_\chi^{\tau-1} + \sum_\tau^\infty\right) r_{nk} C_n(x)\,\Delta(v_n).$$

By (P.3) and (2) the second of the series in (18) is $O(x^{-1})$ together with terms of the form

$$O\left(\frac{t}{x}\left|\sum_\tau^\infty r_{nk}\cos nx\cos nt\right|\right) + O\left(\frac{1}{x}\left|\sum_\tau^\infty \frac{1}{n}\, r_{nk}\cos nx\cos nt\right|\right); \tag{19}$$

summing by parts once more and using both of the estimates for $C_n(x)$, we obtain

$$\sum_{\tau}^{\infty} r_{nk}C_n(x)\,\Delta(v_n) = O(x^{-1}). \tag{20}$$

On the other hand, by (P.4) and (2), the first of the series in (18) is of the form

$$O\left(\frac{1}{x}\sum_{x}^{\tau-1}\{t+n^{-2}\}\right)+O\left(\frac{1}{x}\left|\sum_{x}^{\tau-1}\frac{1}{n}r_{nk}\cos nx\, v_n(t)\right|\right)$$
$$= O(x^{-1})+O\left(\frac{1}{x}\left|\sum_{x}^{\tau-1}\frac{1}{n}C_n(x)\,\Delta(r_{nk}v_n)\right|\right).$$

If now (P.4) is used again, we deduce that

$$\sum_{x}^{\tau-1} r_{nk}C_n(x)\,\Delta(v_n) = O(x^{-1})$$

which together with (20) gives

$$J_{23} = O\left(\frac{1}{x}\int_0^{x/2}|\phi|\right) = O(\phi^*(x)).$$

If the last part of the deduction of the estimate of J_{23} is applied to J_{24} it is clear that

$$J_{24} = O\left(\frac{1}{x}\int_0^{x/2}|\phi|\right) = O(\phi^*(x))$$

also. Consequently,

$$m\left\{x\in\left(0,\frac{\pi}{2}\right) : \sup_k |I_2| > s\right\} \leqq \frac{\text{const}}{s^p}(\|\phi\|_p)^p.$$

(iii) $2x<t\leqq\pi/2$. A proof similar to the one given in part (ii) with the roles of x, t reversed shows that

$$m\left\{x\in\left(0,\frac{\pi}{2}\right) : \sup_k |I_3| > s\right\} \leqq \frac{\text{const}}{s^p}(\|\phi\|_p)^p$$

where

$$I_3 = \int_{2x}^{\pi/2}\phi(t)\left\{\sum_n r_{nk}u_n(x)v_n(t)\right\}dt.$$

Stage 2. Now suppose x, t range over any of the intervals $(0, \pi/2)$, $(\pi/2, \pi)$. By suitable changes of variables using (P.5) the proof can always be reduced to that of Stage 1 (or a simpler version). Suppose, for instance, $t\in(\pi/2,\pi)$, $x\in(0,\pi/2)$ and set $t=\pi-\eta$. Then

$$\int_{\pi/2}^{\pi}\phi(\tau)\left\{\sum_n r_{nk}u_n(x)v_n(t)\right\}dt = O(1)\left(\int_{\pi/2}^{\pi}|\phi(t)|\,dt\right)$$
$$+\int_0^{\pi/2}\phi(\pi-\eta)\left\{\sum_n(-1)^n r_{nk}u_n(x)V_n(\eta)\right\}d\eta$$

for some sequence $\{V_n\}$ satisfying (P.1), ..., (P.4) and related to $\{v_n\}$ by (P.5).

Obviously, if Lemma 1.3 is used instead of Lemma 1.2 the proof of Stage 1 can be repeated. The remaining cases can be taken care of likewise. Consequently, (13) holds for some suitable constant A_p and so Theorem 1 has been established.

Proof of Theorem 2. If $f \to S^*f$ is of weak type (1, 1) then Theorem 2 can be proved in exactly the same way as Theorem 1 except for one point—the integration by parts arguments used to deal with terms of the form (14) (cf., in particular, (15), (17)). Here the property

$$\left\|\frac{1}{x}\int_0^x \phi(t)\,dt\right\|_1 \leqq \text{const}\,\{\|\phi \log^+ |\phi|\,\|_1+1\}$$

and the corresponding result for $\int_x^\pi (\phi(t)/t)\,dt$ is required. Notice also that

$$\|\phi\|_1 \leqq \text{const}\,\{\|\phi \log^+ |\phi|\,\|_1+1\}.$$

Proof of Theorem 3. Suppose now that $f \to S^*f$ is of weak type (1, 1). The stronger condition (P.2)′ is utilized so that the integration by parts argument for (14) may be avoided. Since this and the corresponding estimates in the various cases of Stage 2 are the only parts of the proof of Theorem 1 which are not valid for $p=1$, the proof of Theorem 3 will then be complete.

In J_{12} replace $u_n(x)$, $v_n(t)$ by their asymptotic estimates (P.2)′. Then the only terms causing new complications are series of the form

$$\sum_{\chi}^{\infty} \frac{1}{nx} r_{nk} \sin(\lambda_n x+\beta)\cos(\lambda_n t+\beta) \tag{21}$$

and

$$\sum_{\chi}^{\infty} \frac{1}{nt} r_{nk} \sin(\lambda_n t+\beta)\cos(\lambda_n x+\beta) \tag{22}$$

since also $u_n = v_n$. It is enough to consider (21). Now, the series in (21) is

$$\frac{1}{2}\sum_{\chi}^{\infty}\frac{1}{nx} r_{nk}\{\sin\lambda_n(x-t)-\sin(\lambda_n(x+t)+2\beta)\} = S_1+S_2.$$

Since $\lambda_n = n+a$ for all n, S_1 can be written as

$$\frac{1}{2}\sum_{\chi}^{\infty}\frac{1}{nx} r_{nk}\{\sin n(x-t)\cos a(x-t)+\cos n(x-t)\sin a(x-t)\} = S_{11}+S_{12}.$$

Summation by parts in S_{12} gives

$$S_{12} = O\left(\left|\frac{\sin a(x-t)}{\sin \frac{1}{2}(x-t)}\right|\right) = O(1)$$

uniformly in k. On the other hand, the usual argument to establish the boundedness of $\sum_1^\infty (\sin nx)/n$ (cf. Boas [6, p. 11]) shows that $S_{11}=O(x^{-1})$ uniformly in k and so $S_1 = O(x^{-1})$. Since summation by parts also gives $S_2 = O(x^{-1})$, in this case

$J_{12} = O(\phi^*(x))$ uniformly in k. Thus the integration by parts in Stage 1 has been avoided.

In the various cases of Stage 2 similar, though not identical, arguments can be used. Hence in all cases we have avoided use of integration by parts.

Proof of Theorem 4. The only modification required in the proof of Theorem 3 is the replacing of "weak type" inequalities with the corresponding "strong type" inequalities.

1.5. REMARKS. (i) Obviously (P.2), ..., (P.5) or (P.2)′, (P.3), ..., (P.5) need only be satisfied for all sufficiently large n, say $n > N$, although, of course, the value of the constant A_p in (13) or A_1 in the equivalent expression for Theorems 2–4 then depends on N. (ii) Basically (P.2)′ has to be applied instead of (P.2) to derive the weak type (1, 1) result because $\sum_1^\infty (\sin nx)/n$ is bounded while $\sum_1^\infty (\cos nx)/n$ is $O(x^{-1})$.

2. **Applications.** The weak type or strong type properties of the operator $f \to S^*f$ are most easily checked, perhaps, by introducing the kernel $L_k(x, t)$ and its conjugate $\tilde{L}_k(x, t)$,

$$L_k(x, t) = \sum_{-\infty}^{\infty} r_{nk} e^{inx} e^{-int}, \qquad \tilde{L}_k(x, t) = \sum_{-\infty}^{\infty} (-i) \operatorname{sgn}(n) r_{nk} e^{inx} e^{-int},$$

where $r_{-nk} = r_{nk}$ and $\operatorname{sgn}(0) = 0$. Since then

$$K_k(x, t) = \tfrac{1}{2}\{L_k(x, t) + i\tilde{L}_k(x, t)\} + \tfrac{1}{2} r_{0k},$$

the Riesz theorem ensures that $f \to S^*f$ is of weak type (p, p) if $1 < p < \infty$ and

$$f \to L^*f = \sup_k \left| \int_{-\pi}^{\pi} f(t) L_k(x, t)\, dt \right|$$

is of weak type (p, p) with respect to $L^p(-\pi, \pi)$. In the case $p = 1$, $f \to S^*f$ will be of weak type (1, 1) if, with obvious notation, both $f \to L^*f$ and $\tilde{L}^*f$ are of weak type (1, 1) with respect to $L^1(-\pi, \pi)$. From now on $\{u_n\}$ will denote a complete(5) orthonormal sequence on $(0, \pi)$ satisfying (P.1), ..., (P.5) though (P.2), ..., (P.5) need hold only for $n > N$; obviously the sequence $\{(2/\pi)^{1/2} \cos nx\}$ is one such sequence.

We shall apply Theorem 1 first to the Poisson kernel; the fact that here k is a continuous parameter causes no problems. Suppose first that $u_0 \not\equiv 0$.

THEOREM A. *Let $1 < p < \infty$ and let $\{a_n\}$ be a sequence of real numbers. Then $f \sim \sum_n a_n \cos nx$ for some $f \in L^p(0, \pi)$ if and only if $\phi \sim \sum_n a_n u_n(x)$ for some $\phi \in L^p(0, \pi)$. Furthermore,*

$$C_p \|f\|_p \leqq \|\phi\|_p \leqq C_p' \|f\|_p, \tag{23}$$

C_p, C_p' being constants independent of $\{a_n\}$.

(5) In the sense that linear combinations are dense in $L^p(0, \pi)$, $1 < p < \infty$.

Proof. It is well known that, when $L_k(x, t)$ is the Poisson kernel, the operator $f \to L^*f$ is of strong type (p, p), $1<p<\infty$ (cf. Zygmund [24, Vol. I, p. 155]). Theorem 1 in its "strong type" form can thus be applied.

Let $\{a_n\}=\{\hat{f}(n)\}$ be the Fourier-cosine coefficients of some $f \in L_p(0, \pi)$:

$$\hat{f}(n) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\pi f(t) \cos nt \, dt.$$

Set $\phi_k(x)=\sum_n r_{nk}\hat{f}(n)u_n(x)$. By Theorem 1 with $\{v_n(t)\}=\{(2/\pi)^{1/2} \cos nt\}$, the family $\{\phi_k\}$ is a set uniformly bounded in norm in $L^p(0, \pi)$ and so has a weak*-cluster point $\phi \in L^p(0, \pi)$. Now, if C'_p is the constant of Theorem 1 corresponding to the pair $\{u_n(x)\}$, $\{(2/\pi)^{1/2} \cos nt\}$, then $\|\phi\|_p \leqq C'_p\|f\|_p$. On the other hand, since $u_n \in L^\infty(0, \pi)$ and $\lim_{k\to\infty} r_{nk}=1$, clearly $\int_0^\pi \phi(x)u_n(x)\,dx=\hat{f}(n)$, i.e., $\phi \sim \sum_n a_n u_n(x)$.

To obtain the other half of the theorem all we need do is reverse the roles of $\{u_n\}$, $\{(2/\pi)^{1/2} \cos nt\}$.

When $u_0 \equiv 0$, a similar proof gives

THEOREM B. *Let $1<p<\infty$ and let $\{a_n\}$ be a sequence of real numbers, $a_0=0$. Then $f\sim \sum_n a_n \sin nx$ for some $f \in L^p(0, \pi)$ if and only if $\phi \sim \sum_n a_n u_n(x)$ for some $\phi \in L^p(0, \pi)$. Furthermore, $C_p\|f\|_p \leqq \|\phi\|_p \leqq C'_p\|f\|_p$ for constants C_p, C'_p.*

A sequence $\{\lambda_n\}$ of real numbers is said to be an L^p-multiplier for $\{(2/\pi)^{1/2} \cos nx\}$ if

(i) $\sum_n \lambda_n\hat{f}(n) \cos nx$ is the Fourier-cosine expansion of a function $F \in L^p(0, \pi)$ whenever $f\sim\sum_n \hat{f}(n) \cos nx$ belongs to $L^p(0, \pi)$, and

(ii) for some constant A_p, independent of f, $\|F\|_p \leqq A_p\|f\|_p$.

We make the same definition with respect to any other orthonormal sequence $\{u_n\}$. Theorems A, B give immediately

THEOREM C. *Let $1<p<\infty$ and let $\{\lambda_n\}$ be a sequence of real numbers. Then $\{\lambda_n\}$ is an L^p-multiplier for $\{(2/\pi)^{1/2} \cos nx\}$ if and only if it is an L^p-multiplier for $\{u_n(x)\}$.*

For each positive integer n denote by $S_n(\phi, x)$ the nth-partial sum

$$S_n(\phi, x) = \sum_0^n \hat{\phi}(m)u_m(x), \qquad \hat{\phi}(m) = \int_0^\pi \phi(t)u_m(t)\,dt. \tag{24}$$

THEOREM D. *Let $1<p<\infty$. Then $S_n(\phi, x)$ converges in norm and*

$$\sup_n \|S_n(\phi, x)\|_p \leqq C_p\|\phi\|_p, \qquad \phi \in L^p(0, \pi), \tag{25}$$

for some constant C_p.

Proof. In view of the completeness of $\{u_n\}$ it is enough to establish (25). Now, if $\phi\sim\sum_n \hat{\phi}(n)u_n(x)$, $f\sim\sum_n \hat{\phi}(n) \cos nx$, the partial sums

$$\sigma_n(f, x) = \sum_0^n \hat{\phi}(m) \cos mx$$

satisfy

$$\sup_n \|\sigma_n(f, x)\|_p \leqq \text{const } \|f\|_p$$

by the Riesz theorem. Application of both parts of (23) then gives (25).

Now suppose that $\lim_{k\to\infty} r_{nk}$ exists for each n. If the operator $f \to S^*f$ is of weak type (p, p) for some p, $1 \leqq p < \infty$, the usual type of argument shows that $\lim_{k\to\infty} S_k^* f$ exists a.e. on $(-\pi, \pi)$ for each $f \in L^p(-\pi, \pi)$ when

$$S_k f(x) = \int_{-\pi}^{\pi} f(t)\left\{\sum_{n=0}^{\infty} r_{nk} e^{inx} e^{-int}\right\} dt.$$

However, in a remarkable theorem [17, Theorem 1, p. 148] Stein has shown that, under certain circumstances, the converse is true also, i.e., convergence a.e. implies weak type. Thus, when $L_k f$, $T_k \phi$ are defined by

$$L_k f = \sum_{-\infty}^{\infty} r_{nk} \hat{f}(n) e^{inx}, \qquad T_k \phi = \sum_{0}^{\infty} r_{nk} \hat{\phi}(n) u_n(x)$$

where $\hat{f}(n) = \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$, $\hat{\phi}(n) = \int_0^{\pi} \phi(t) u_n(t)\, dt$ we deduce from Stein's theorem a result of equiconvergence or equisummability character:

THEOREM E. *Let* $1 < p \leqq 2$. *If* $\lim_{k\to\infty} L_k f$ *exists a.e. on* $(-\pi, \pi)$ *for each* $f \in L^p(-\pi, \pi)$, *then*

(i) *the operator* $\phi \to \mathcal{T}^*\phi = \sup_k |T_k \phi|$ *is of weak type* (p, p),

(ii) $\lim_{k\to\infty} T_k \phi$ *exists a.e. on* $(0, \pi)$,

whenever $\phi \in L^p(0, \pi)$.

Proof. Clearly the operators $f \to L_k f$ satisfy the hypotheses of Stein's theorem. Part (i) of Theorem E then follows immediately; part (ii) now follows from (i) using, for example, Theorem D in the usual way (cf. Edwards [9, Vol. II, p. 193]).

REMARK. If in addition (P.2)′ is satisfied and $\lim_{k\to\infty} L_k f$, $\lim_{k\to\infty} \tilde{L}_k f$ exist a.e. on $(-\pi, \pi)$ for each $f \in L^1(-\pi, \pi)$ then: For $\phi \in L^1(0, \pi)$

(i) the operator $\phi \to \mathcal{T}^*\phi$ is of weak type (1, 1),

(ii) $\lim_{k\to\infty} T_k \phi$ exists a.e. on $(0, \pi)$.

To deduce (ii) from (i) we could, for instance, use the density of L^p-functions in $L^1(0, \pi)$, $p > 1$, together with Theorem E (ii).

Our final application is the extension of the Littlewood-Paley and Carleson-Hunt theorems to orthogonal expansions with respect to $\{u_n\}$.

THEOREM F. *Let* $1 < p < \infty$ *and let* $S_k(\phi, x)$ *be defined by* (24). *Then*

(i) *if* $\{\lambda_k\}$ *is a lacunary sequence of positive integers*

$$\left\|\sup_k |S_{\lambda_k}(\phi, x)|\right\|_p \leqq A_p \|\phi\|_p,$$

(ii) *there is a constant* $B_p = O((p-1)^{-2})$ *such that*

$$\left\|\sup_k |S_k(\phi, x)|\right\|_p \leqq B_p \|\phi\|_p,$$

for all $\phi \in L^p(0, \pi)$.

Proof. Clearly the coefficients of the Dirichlet kernel satisfy I, II (strictly speaking, the coefficients corresponding to nonnegative integers). Parts (i), (ii) of Theorem F are now an immediate consequence of the corresponding theorems for Fourier series (cf. Zygmund [24, Vol. II, p. 231], Carleson [7] and Hunt [12]) and Theorem 1 in its "strong type" form (or its weak type form together with the Marcinkiewicz interpolation theorem).

2.1. REMARKS. (i) If the operator $f \to L^*f$ is of strong type (p, p) for some p, $1 \leqq p < \infty$ then, for each k, the sequence $\{r_{nk}\}$ is a multiplier for $L^p(-\pi, \pi)$ in the usual sense of Fourier series; in particular

$$\left\| \sum_{-\infty}^{\infty} r_{nk} \hat{f}(n) e^{inx} \right\|_p \leqq A_p \|f\|_p$$

where A_p is independent of k. Since $\{r_{nk}\}$ is also a multiplier for $L^q(-\pi, \pi)$, $1/q + 1/p = 1$, with the same norm A_p we conclude that, for each k, $\{r_{nk}\}$ is a multiplier for $L^2(-\pi, \pi)$ with norm independent of k. Since the norm of any such multiplier on $L^2(-\pi, \pi)$ is merely the l^∞-norm of $\{r_{nk}\}$ it is clear that the uniform boundedness condition I is necessary if $f \to L^*f$ is to be of strong type (p, p). Similarly, by the Marcinkiewicz interpolation theorem, if $f \to L^*f$ is to be of weak type (p, p) for two distinct values of p, condition I is necessary.

The assumption that each $\{r_{nk}\}$ is absolutely summable is very natural since we are usually concerned with the cases when the $\{r_{nk}\}$ arise from a summability method (cf. Zygmund [24, Vol. I, p. 84]). In this case, condition II is a necessary (though not sufficient) condition for the summability method to be regular [24, Vol. I, p. 75].

(ii) The full strength of Theorem 1 is not required in the proof of Theorem A. As is well known, any sequence $\{\lambda_n\}$ of bounded variation is multiplier for Fourier series of functions in $L^p(-\pi, \pi)$, $1 < p < \infty$, and

$$\left\| \sum_n \lambda_n \hat{f}(n) e^{inx} \right\|_p \leqq A_p \left(\sum_n |\Delta(\lambda_n)| \right) \|f\|_p, \qquad A_p \text{ constant.}$$

Consequently, under conditions I, II, automatically,

$$\sup_k \left\| \int_{-\pi}^{\pi} f(t) \left\{ \sum_{-\infty}^{\infty} r_{nk} e^{inx} e^{-int} \right\} dt \right\|_p \leqq B_p \|\phi\|_p.$$

The proof of Theorem 1 then gives

$$\sup_k \left\| \int_0^{\pi} \phi(t) \left\{ \sum_k r_{nk} u_n(x) v_n(t) \right\} dt \right\|_p \leqq C_p \|\phi\|_p$$

with C_p independent of k. Theorems A, B still follow from this weakened form of Theorem 1.

(iii) Result (ii) quoted in the introduction is a combination of Theorems A, F (ii).

3. **Examples.** We shall now show, albeit briefly, that conditions (P.1), ..., (P.5) and (P.2)′ are satisfied by a large class of sequences on $(0, \pi)$ including the classical

orthonormal sequences: Fourier-Bessel functions, Fourier-Dini functions, ultraspherical polynomials and Jacobi polynomials. The proofs will be deliberately sketchy because elsewhere [11] these conditions are established for two large classes of eigenfunctions of singular and nonsingular Sturm-Liouville systems, allowing a more unified proof. Many of the previously known results (as well as new ones) then appear as special cases of the theorems in §2.

(i) *Perturbed cosine and sine functions.* If a, α and β are fixed constants, consider

$$\phi_n(x) = \cos(\lambda_n x + a), \qquad \psi_n(x) = \sin(\lambda_n x + a) \tag{26}$$

where

$$\lambda_n = n + \alpha + \beta/n + O(n^{-2}), \qquad n > N_0. \tag{27}$$

Though, in general, neither orthogonal nor normalized, the sequences $\{\phi_n\}$, $\{\psi_n\}$ satisfy (P.1), (P.2)$'$, . . ., (P.5) as is almost trivial to check.

(ii) *Fourier-Bessel functions.* If $J_\alpha(x)$ is the Bessel function of the first kind of order α, $\alpha \geqq -\frac{1}{2}$, and $\lambda_1\pi, \lambda_2\pi, \ldots$ are the positive roots of $J_\alpha(x)=0$ (indexed in increasing order), then the Fourier-Bessel functions $j_n^\alpha(x) = \sigma_n^\alpha x^{1/2} J_\alpha(\lambda_n x)$ form an orthonormal sequence on $(0, \pi)$ the normalizing constants σ_n^α being given by

$$\sigma_n^\alpha = (2^{1/2}/\pi)|J_{\alpha+1}(\lambda_n \pi)|^{-1},$$

(Watson [22, p. 576]). We set $j_0^\alpha \equiv 0$. The roots $\{\lambda_n\pi\}$ can be estimated asymptotically: for some integer l and all $n > N_0$,

$$\lambda_n\pi = (n - l + \alpha/2 - 1/4)\pi - (\alpha^2 - 1/4)/2n\pi + O(n^{-3}), \tag{28}$$

[22, p. 506]. The integer l arises in (28) because the asymptotic expression given by Watson for the zeros of $J_\alpha(x)$ is valid only for $x > N_0$, say, and we cannot presume that the first zero encountered in $[N_0, \infty)$ is the "N_0th zero" since there may be a large number of zeros close to 0 (cf. comments [22, p. 494]). The specific value of l is quite immaterial for our purposes. Using the classical 2-term asymptotic estimate for J_α valid for $x > 1$

$$x^{1/2} J_\alpha(x) = \left(\frac{2}{\pi}\right)^{1/2} \left\{ \cos\left(x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{\alpha^2 - 1/4}{2x} \sin\left(x - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + R_\alpha(x) \right\} \tag{29}$$

where $R_\alpha(x) = O(x^{-2})$ [22, p. 199] and the properties

$$J_\alpha(x) = O(x^\alpha), \qquad x \to 0+, \qquad J_\alpha(x) = O(x^{-1/2}), \qquad x \to \infty,$$

we can soon check that

$$j_n^\alpha(x) = (\lambda_n x)^{1/2} J_\alpha(\lambda_n x) + O(n^{-2}), \qquad n > N_0. \tag{30}$$

Since the first N_0 terms clearly are bounded, property (P.1) for $\{j_n^\alpha\}$ follows immediately; (P.2)$'$ also follows from (30) after substitution of (28), (29). Since there is an explicit expression for the remainder term $R_\alpha(x)$ in (29) (cf. [22, pp. 196, 197]), property (P.3) can be checked by a direct calculation of $\Delta(R_\alpha(\lambda_n x))$. For large values of α, in fact when $\alpha \geqq 5/2$, this method cannot be applied immediately since then the integrals involved do not necessarily converge. However, this difficulty can be avoided by taking more terms in the asymptotic estimate (29) exactly as Watson does in estimating $R_\alpha(x)$ [22, pp. 197, 198].

An easy mean-value theorem argument using the recurrence relation

$$J'_\alpha(x) = -J_{\alpha+1}(x)+(\alpha/x)J_\alpha(x) \tag{31}$$

gives (P.4) while, finally, (P.5) in its strong form follows almost immediately from (28), (29) and (30). Theorems 1–4 and Theorems A–F can thus be applied to Fourier-Bessel series.

(iii) *Fourier-Dini functions.* If $\gamma_1\pi, \gamma_2\pi, \ldots$ are the positive roots of the equation $J_\alpha(x)+HxJ'_\alpha(x)=0$, H fixed and real, the Fourier-Dini functions

$$k_n^\alpha(x)=\tau_n^\alpha x^{1/2}J_\alpha(\gamma_n x), \qquad n = 1, 2, \ldots,$$

form an orthonormal sequence on $(0, \pi)$, $\alpha \geqq -\frac{1}{2}$, the normalizing constant τ_n^α being given by

$$\tau_n^\alpha = (2^{1/2}/\pi)\{(J'_\alpha(\gamma_n\pi))^2+(1-(\alpha/\gamma_n\pi)^2)(J_\alpha(\gamma_n\pi))^2\}^{-1/2}$$

[22, p. 577](6). Using the method of McMahon [13] we can soon show that, for some integer l_1 and all $n > N_0$,

$$\gamma_n\pi = (n-l_1+\alpha/2+1/4)\pi-((\alpha^2-1/4)\beta-2)/2n\beta\pi+O(n^{-3})$$

with β a constant depending only on H. The properties (P.1), . . . , (P.5) can now be derived exactly as for Fourier-Bessel functions.

Only convergence in norm (Theorem D) for Fourier-Bessel and Fourier-Dini series appears to have been discussed before (cf. Wing [23]), although Muckenhoupt and Stein [14, pp. 90–91], do suggest that many of the results in §2 will hold for these series. Watson [22, Chapter 18], Titchmarsh [21, p. 73] and Stone [18] discuss equiconvergence and equisummability results (cf. Theorem E).

(iv) *Ultraspherical and Jacobi polynomials.* The ultraspherical polynomials $P_n^\lambda(\cos x)$ of index λ and degree n are defined by

$$(1-2z\cos x+z^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(\cos x)z^n, \qquad \lambda > 0.$$

For fixed λ the functions

$$P_n^\lambda(x) = \gamma_n^\lambda P_n^\lambda(\cos x)(\sin x)^\lambda,$$

(6) To avoid the troublesome modifications required when $H+\alpha \leqq 0$ we assume, for convenience, that $H+\alpha>0$, (Watson [22, p. 597]).

form a sequence orthonormal on $(0, \pi)$ when

$$(\gamma_n^\lambda)^2 = \frac{n!\,(n+\lambda)\Gamma(\lambda)\Gamma(2\lambda)}{\Gamma(n+2\lambda)\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})},$$

(cf. Szegö [19, p. 82]). Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$, $-1<y<1$, of degree n and order (α, β) with $\alpha, \beta > -1$ are defined by

$$(1-y)^\alpha(1+y)^\beta P_n^{(\alpha,\beta)}(y) = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dy}\right)^n[(1-y)^{n+\alpha}(1+y)^{n+\beta}].$$

When normalizing constants $t_n^{(\alpha,\beta)}$ are defined by

$$t_n^{(\alpha,\beta)} = \left\{\frac{n!\,(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right\}^{1/2},$$

the sequence

$$q_n^{(\alpha,\beta)}(x) = t_n^{(\alpha,\beta)}P_n^{(\alpha,\beta)}(\cos x)(\sin x/2)^{\alpha+1/2}(\cos x/2)^{\beta+1/2}2^{(\alpha+\beta+1)/2}$$

is orthonormal on $(0, \pi)$, [18, p. 67]. Since ultraspherical polynomials are a special case of Jacobi polynomials we shall establish (P.1), ..., (P.5) and (P.2)′ only for the latter.

Property (P.1), $\alpha, \beta \geqq -\frac{1}{2}$, is well known (cf., for instance, Pollard [15, p. 356]). For (P.2) we could use the formula of Darboux [8] quoted by Askey [2, II(4)] or, in the case of ultraspherical polynomials, a result of Szegö [19] quoted by Askey-Wainger [5, p. 400]. However, by repeating the Liouville-Stekloff method given by Szegö [20, p. 208] and Rau [16] we obtain a 2-term Hilb-type asymptotic estimate which is more than sufficient for our purposes. Szegö apparently discusses such an expansion in his paper [19] (cf. comments [20, pp. 200, 212]) but, as Askey-Wainger say in [5, p. 400], Szegö's paper [19] is "relatively inaccessible".

3.1. THEOREM. *Let* $\alpha, \beta \geqq -\frac{1}{2}$ *and* $N = n+\frac{1}{2}(\alpha+\beta+1)$. *Then*

(i) *in* $(0, 1/n]$

$$q_n^{(\alpha,\beta)}(x) = (Nx)^{1/2}J_\alpha(Nx)+O(n^{-2}),$$

(ii) *there exists* $X \in L^\infty(0, \pi/2)$ *such that, in* $(1/n, \pi/2)$,

$$q_n^{(\alpha,\beta)}(x) = (Nx)^{1/2}J_\alpha(Nx)+(1/n)X(x)(Nx)^{1/2}Y_\alpha(Nx)+O(n^{-2}),$$

where both error terms are uniform and Y_α *is the Bessel function of the second kind of order* α.

A detailed proof for a similar situation is given in [11]. There is a corresponding pair of estimates for $(\pi/2, \pi)$ or, alternatively, the property:

$$P_n^{(\alpha,\beta)}(\cos x) = (-1)^n P_n^{(\beta,\alpha)}(\cos(\pi-x))$$

could be used (Szegö [20, p. 59]). Notice too that (P.1) is a consequence of the theorem. If we now use the asymptotic expansion (29) and the corresponding expression for Y_α (Watson [22, p. 199]), property (P.2)′ follows immediately. Also, the same argument as for Fourier-Bessel functions gives (P.3). Property (P.4) follows from part (i) of Theorem 3.1 by a mean-value argument together with (31). Alternatively, (P.3) and (P.4) could be derived from the "differentiation" formula for Jacobi polynomials (Szegö [20, (4.21.7), (4.5.7)]) and the asymptotic estimates of Theorem 3.1. Finally, (P.5) in its strong form is an immediate consequence of the property $q_n^{(\alpha,\beta)}(\pi-x)=(-1)^n q_n^{(\beta,\alpha)}(x)$ of Jacobi polynomials quoted already. Thus Theorems 1–4 and Theorems A–F are true also for ultraspherical and Jacobi polynomials, $\lambda>0$, $\alpha, \beta \geqq -\frac{1}{2}$.

Theorem A has been established for ultraspherical polynomials by Askey-Wainger [4] using a more difficult proof and for Jacobi polynomials by Askey [2, Theorem 1] by a method on which the proof of Theorem 1 given here is based. In both [2] and [4] only the Poisson kernel is used; also, Theorem D is deduced from Theorem A in both of these papers for the orthonormal sequences in question. Earlier, Pollard [15] had given a direct proof of Theorem D for Jacobi (and more general) polynomials. Equiconvergence and equisummability results are discussed in Szegö [20, Chapter IX].

Theorem F(i) and the Marcinkiewicz multiplier theorem were established for ultraspherical polynomials by Muckenhoupt-Stein [14, Chapter II] using ingenious arguments paralleling the proofs of the corresponding results in Fourier series. A direct method requiring less sophisticated asymptotic estimates was used in [10] to prove Theorem F(ii) directly for ultraspherical and Jacobi polynomials.

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